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Classification of n-qubit states with minimum orbit dimension

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Abstract

The group of local unitary transformations acts on the space of n-qubit pure states, decomposing it into orbits. In a previous paper we proved that a product of singlet states (together with an unentangled qubit for a system with an odd number of qubits) achieves the smallest possible orbit dimension, equal to $3n/2$ for n even and $(3n+1)/2$ for n odd, where n is the number of qubits. In this paper we show that any state with minimum orbit dimension must be of this form, and furthermore, such states are classified up to local unitary equivalence by the sets of pairs of qubits entangled in singlets.

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1. Introduction

Quantum entanglement has been identified as an important resource for quantum computing, quantum communication and other applications [1, 2]. A fundamental theoretical problem is to understand the types of entanglement that composite quantum systems can achieve.

Defining entanglement types as equivalence classes of quantum states under local unitary (LU) equivalence is perhaps the most natural way to proceed in classifying entanglement [3, 4]. The group of LU transformations acts on the space of quantum states, partitioning it into LU orbits. Each orbit is a collection of locally equivalent quantum states that forms a differentiable manifold with a certain dimension.

The classification of entanglement types has turned out to be a difficult problem. Most of the progress in understanding multipartite entanglement has been for systems of only two or three qubits [5–7]. Few results exist concerning the classification of n-qubit entanglement types for arbitrary n.

A promising approach to the difficult problem of characterizing entanglement types is to break the problem into two parts. First, identify the possible dimensions of LU orbits in the state space. Then, identify the orbits that have each possible dimension. In [8], the present authors addressed the first part of this program by identifying the allowable range of orbit dimensions for n-qubit quantum states to be $3n/2$ to $3n$ for even n and $(3n+1)/2$ to $3n$ for odd n. The present paper begins the second part of the program by completely identifying all orbits (entanglement types) with minimum dimension.

In this paper, we identify all n-qubit entanglement types that have a minimum orbit dimension. States that have the minimum orbit dimension are, in some sense, the 'rarest' quantum states. We show that the only quantum states to achieve the minimum orbit dimension are tensor products of singlet states (with a single unentangled qubit for n odd) and their LU equivalents. This suggests a special role for the 2-qubit singlet state in the theory of n-qubit quantum entanglement.

2. Notation and previous results

In this section we establish notation, some definitions, and state some results from our previous paper [8] needed for the present paper. For the convenience of the reader, we give a list (appendix A) of equation and statement numbers in the present paper with their matching numbers in [8].

Let $|0\rangle, |1\rangle$ denote the standard computational basis for $\mathbb{C}^2$ and write $|i_1 i_2 \ldots i_n\rangle$ for $|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle$ in $(\mathbb{C}^2)^{\otimes n}$. For a multi-index $I = (i_1 i_2 \ldots i_n)$ with $i_k = 0, 1$ for $1 \leqslant k \leqslant n$, we will write $|I\rangle$ to denote $|i_1 i_2 \ldots i_n\rangle$. Let i_k^c denote the bit complement

$$i_k^c = \begin{cases} 0 & \text{if } i_k = 1 \\ 1 & \text{if } i_k = 0 \end{cases}$$

and let I_k denote the multi-index

$$I_k := \left(i_1 i_2 \cdots i_{k-1} i_k^c i_{k+1} \cdots i_n\right)$$

obtained from I by taking the complement of the kth bit for $1 \leqslant k \leqslant n$. Similarly, let I_{kl} denote the multi-index

$$I_{kl} := \left(i_1 i_2 \cdots i_{k-1} i_k^c i_{k+1} \cdots i_{l-1} i_l^c i_{l+1} \cdots i_n\right)$$

obtained from I by taking the complement of the kth and lth bits for $1 \leqslant k < l \leqslant n$.

Let $H = (\mathbb{C}^2)^{\otimes n}$ denote the Hilbert space for a system of n qubits and let $\mathbb{P}(H)$ denote the projectivization of H which is the state space of the system. We take the local unitary group to be $G = \mathrm{SU}(2)^n$. Its Lie algebra $LG = \mathrm{su}(2)^n$ is the space of n-tuples of traceless skew-Hermitian 2×2 matrices. We choose $A = \mathrm{i}\sigma_z = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, $B = \mathrm{i}\sigma_y = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $C = \mathrm{i}\sigma_x = \left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]$ as a basis for su(2), where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices. We define elements A_k, B_k, C_k of LG for $1 \leqslant k \leqslant n$ to have A, B, C, respectively, in the kth coordinate and zero elsewhere.

$$A_k = \left(0, \ldots, 0, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, 0, \ldots, 0\right)$$

$$B_k = \left(0, \ldots, 0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right)$$

$$C_k = \left(0, \ldots, 0, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, 0, \ldots, 0\right).$$

Given a state vector $|\psi\rangle = \sum c_I |I\rangle$, we have the following:

$$A_k|\psi\rangle = \sum_I i(-1)^{i_k} c_I |I\rangle \tag{1}$$

$$B_k|\psi\rangle = \sum_I (-1)^{i_k} c_{I_k}|I\rangle \tag{2}$$

$$C_k|\psi\rangle = \sum_I i c_{I_k}|I\rangle. \tag{3}$$

Given a state vector $|\psi\rangle \in H$, we define the $2^n \times (3n + 1)$ complex matrix M to be the $(3n + 1)$-tuple of column vectors

$$M = (A_1|\psi\rangle, B_1|\psi\rangle, C_1|\psi\rangle, \ldots, A_n|\psi\rangle, B_n|\psi\rangle, C_n|\psi\rangle, -\mathrm{i}|\psi\rangle). \tag{4}$$

We view the matrix M and also its column vectors both as real and also as complex via the standard identification

$$\begin{aligned} &\mathbb{C}^N \leftrightarrow \mathbb{R}^{2N} \\ &(z_1, z_2, \ldots, z_N) \leftrightarrow (a_1, b_1, a_2, b_2, \ldots, a_N, b_N), \end{aligned} \tag{5}$$

where $z_j = a_j + \mathrm{i}b_j$ for $1 \leqslant j \leqslant N$. The real rank of M gives the dimension of the G-orbit of a state.

Proposition 2.1. *Orbit dimension as rank of M. Let $x \in \mathbb{P}(H)$ be a state, let $|\psi\rangle$ be a Hilbert space representative for x, and let M be the associated matrix constructed from the coordinates of $|\psi\rangle$ as described above. Then we have*

$$\mathrm{rank}_{\mathbb{R}} M = \dim \mathcal{O}_x + 1.$$

We denote by $\mathcal{C}$ the set

$$\mathcal{C} = \{A_1|\psi\rangle, B_1|\psi\rangle, C_1|\psi\rangle, \ldots, A_n|\psi\rangle, B_n|\psi\rangle, C_n|\psi\rangle, -\mathrm{i}|\psi\rangle\}$$

of columns of M, and for $1 \leqslant k \leqslant n$ we define the *triple* $T_k \subseteq \mathcal{C}$ to be the set of vectors

$$T_k = \{A_k|\psi\rangle, B_k|\psi\rangle, C_k|\psi\rangle\}. \tag{6}$$

For a subset $\mathcal{B} \subseteq \mathcal{C}$, we write $\langle \mathcal{B} \rangle$ to denote the real span of the column vectors contained in $\mathcal{B}$. For convenience, we write $\left\langle T_{i_1}, T_{i_2}, \ldots, T_{i_r} \right\rangle$ to denote the real span $\left\langle T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_r} \right\rangle$ of a set of triples. We have the following facts about the dimensions of spans of sets of triples.

Proposition 2.2. *Let $T_k = \{A_k|\psi\rangle, B_k|\psi\rangle, C_k|\psi\rangle\}$ be a triple of columns of M. The three vectors in the triple are orthogonal when viewed as real vectors.*

Proposition 2.3. *Main orthogonality proposition. Suppose that*

$$\dim \left\langle T_{j_1}, T_{j_2}, \ldots, T_{j_m} \right\rangle < 3m$$

for some $1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant n$. Then there is a nonempty subset $K \subseteq \{j_1, j_2, \ldots, j_m\}$ containing an even number of elements such that there are two orthogonal vectors $|\zeta_k\rangle, |\eta_k\rangle$ in $\langle T_k \rangle$, both of which are orthogonal to $-\mathrm{i}|\psi\rangle, A_j|\psi\rangle, B_j|\psi\rangle$ and to $C_j|\psi\rangle$ for all $k \in K$, $j \notin K$.

More can be said about a pair of triples which together span fewer than five dimensions.

Lemma 2.4. *Suppose that for some $1 \leqslant l < l' \leqslant n$ we have $A_l|\psi\rangle = A_{l'}|\psi\rangle$ and $C_l|\psi\rangle = C_{l'}|\psi\rangle$. Then $A_k|\psi\rangle, B_k|\psi\rangle$ and $C_k|\psi\rangle$ are each orthogonal to $-\mathrm{i}|\psi\rangle$ and to $A_j|\psi\rangle, B_j|\psi\rangle, C_j|\psi\rangle$ for all $k \in \{l, l'\}$, $j \notin \{l, l'\}$.*

Proposition 2.5. *Generalization of lemma 2.4. Suppose that $\dim\langle T_l, T_{l'} \rangle \leqslant 4$ for some $1 \leqslant l < l' \leqslant n$. Then $A_k|\psi\rangle, B_k|\psi\rangle$ and $C_k|\psi\rangle$ are each orthogonal to $-\mathrm{i}|\psi\rangle$ and to $A_j|\psi\rangle, B_j|\psi\rangle, C_j|\psi\rangle$ for all $k \in \{l, l'\}$, $j \notin \{l, l'\}$.*

We use propositions 2.2, 2.3 and 2.5 to establish the lower bound for the rank of M, and therefore also for orbit dimension.

Proposition 2.6. *Minimum rank of M. Let $|\psi\rangle \in H$ be a state vector, and let M be the matrix associated with $|\psi\rangle$. We have*

$$\operatorname{rank}_{\mathbb{R}} M \geqslant \begin{cases} \dfrac{3n}{2}+1 & n \text{ even} \\ \dfrac{3n+1}{2}+1 & n \text{ odd.} \end{cases}$$

A product of singlets (together with an unentangled qubit for n odd) achieves the lower bound established in proposition 2.6. This establishes the minimum orbit dimension.

Theorem 2.7. *For the local unitary group action on the state space for n qubits, the smallest orbit dimension is*

$$\min\{\dim \mathcal{O}_x : x \in \mathbb{P}(H)\} = \begin{cases} \dfrac{3n}{2} & n \text{ even} \\ \dfrac{3n+1}{2} & n \text{ odd.} \end{cases}$$

This concludes the statements of definitions and results from [8] to be used in the following.

3. Further results on ranks of sets of triples

In this section we develop more facts about ranks of sets of triples of the columns of the matrix M associated with an n-qubit state vector $|\psi\rangle$ as described in the previous section. These include strengthened versions of propositions 2.4, 2.5 and 2.6. We begin with a general fact about local unitary invariance.

Proposition 3.1. *The dimension of the span of a union of triples with or without the rightmost column $-\mathrm{i}|\psi\rangle$ is a local unitary invariant.*

Proof. We prove the proposition for the case 'with the rightmost column'. The same proof works for the case 'without the rightmost column' by making the obvious changes.

Let $T_{j_1}, T_{j_2}, \ldots, T_{j_m}$ be a set of triples of columns of the matrix M associated with the state vector $|\psi\rangle$. Let $U = \prod_{i=1}^n U_i$ be an element of the local unitary group and let M' with triples $T'_k = \{A_k U|\psi\rangle, B_k U|\psi\rangle, C_k U|\psi\rangle\}$ be associated with the state $|\psi'\rangle = U|\psi\rangle$. Since U is unitary, the dimension of the span

$$\dim\big\langle T'_{j_1} \cup T'_{j_2} \cup \ldots \cup T'_{j_m} \cup \{-\mathrm{i}|\psi'\rangle\}\big\rangle$$

is equal to the dimension of the span of the set

$$(*) \quad \bigcup_{i=1}^{m} \big\{U^\dagger A_{j_i} U|\psi\rangle, U^\dagger B_{j_i} U|\psi\rangle, U^\dagger C_{j_i} U|\psi\rangle\big\} \cup \{-\mathrm{i}|\psi\rangle\}.$$

Observe that

$$\begin{aligned} U^\dagger A_k U &= \big(0, \ldots, 0, U_k^\dagger A U_k, 0, \ldots, 0\big) = \big(0, \ldots, 0, \mathrm{Ad}\big(U_k^\dagger\big)(A), 0, \ldots, 0\big) \\ U^\dagger B_k U &= \big(0, \ldots, 0, U_k^\dagger B U_k, 0, \ldots, 0\big) = \big(0, \ldots, 0, \mathrm{Ad}\big(U_k^\dagger\big)(B), 0, \ldots, 0\big) \\ U^\dagger C_k U &= \big(0, \ldots, 0, U_k^\dagger C U_k, 0, \ldots, 0\big) = \big(0, \ldots, 0, \mathrm{Ad}\big(U_k^\dagger\big)(C), 0, \ldots, 0\big) \end{aligned}$$

where the zeros occur in all but the kth coordinate and $\mathrm{Ad}: SU(2) \to SO(su(2))$ denotes the adjoint representation of $SU(2)$ on its Lie Algebra. It follows that the span of the set $(*)$ lies inside the span of the set

$$(**) \qquad \bigcup_{i=1}^{m} \{A_{j_i}|\psi\rangle, B_{j_i}|\psi\rangle, C_{j_i}|\psi\rangle\} \cup \{-\mathrm{i}|\psi\rangle\}.$$

and hence that

$$\dim\langle T'_{j_1} \cup T'_{j_2} \cup \ldots \cup T'_{j_m} \cup \{-\mathrm{i}|\psi'\rangle\}\rangle \leqslant \dim\langle T_{j_1} \cup T_{j_2} \cup \ldots \cup T_{j_m} \cup \{-\mathrm{i}|\psi\rangle\}\rangle.$$

Reversing the roles of $|\psi\rangle$ and $|\psi'\rangle$ yields that the span of the set $(**)$ lies inside the span of the set $(*)$, and hence that

$$\dim\langle T'_{j_1} \cup T'_{j_2} \cup \ldots \cup T'_{j_m} \cup \{-\mathrm{i}|\psi'\rangle\}\rangle \geqslant \dim\langle T_{j_1} \cup T_{j_2} \cup \ldots \cup T_{j_m} \cup \{-\mathrm{i}|\psi\rangle\}\rangle.$$

This concludes the proof. □

Next we consider the case of two triples which together span five dimensions.

Proposition 3.2. *Suppose that* $\dim\langle T_l, T_{l'}\rangle = 5$ *for some* $1 \leqslant l < l' \leqslant n$. *Then there are four independent vectors* $|\zeta_l\rangle, |\eta_l\rangle \in T_l$, $|\zeta_{l'}\rangle, |\eta_{l'}\rangle \in T_{l'}$, *which are orthogonal to* $-\mathrm{i}|\psi\rangle$ *and to* $A_j|\psi\rangle, B_j|\psi\rangle, C_j|\psi\rangle$ *for all* $j \notin \{l, l'\}$.

Proof. Applying the main orthogonality proposition 2.3, the only new claim made in the statement of 3.2 is that the vectors $|\zeta_k\rangle, |\eta_k\rangle \in T_k$, $k \in \{l, l'\}$ are independent. In the proof of proposition 2.3 there is a unitary transformation $U: H \to H$, such that

$$U|\zeta_k\rangle = B_k U|\psi\rangle \qquad U|\eta_k\rangle = C_k U|\psi\rangle$$

for $k \in \{l, l'\}$ and $A_l U|\psi\rangle$ is collinear with $A_{l'} U|\psi\rangle$. Since $\dim\langle T_l, T_{l'}\rangle = 5$, the span of $\{A_k U|\psi\rangle, B_k U|\psi\rangle, C_k U|\psi\rangle : k = l, l'\}$ is also five-dimensional by proposition 3.1. Thus the collinearity of $A_l U|\psi\rangle$ and $A_{l'} U|\psi\rangle$ implies that $U|\zeta_l\rangle, U|\eta_l\rangle, U|\zeta_{l'}\rangle$ and $U|\eta_{l'}\rangle$ are independent, and therefore that $|\zeta_l\rangle, |\eta_l\rangle, |\zeta_{l'}\rangle$ and $|\eta_{l'}\rangle$ are independent. □

Next, a small observation proves a stronger version of lemma 2.4.

Lemma 3.3. *Suppose that for some* $1 \leqslant l < l' \leqslant n$, *we have* $A_l|\psi\rangle = A_{l'}|\psi\rangle$ *and* $C_l|\psi\rangle = C_{l'}|\psi\rangle$. *Then* $B_l|\psi\rangle = -B_{l'}|\psi\rangle$, *the dimension of* $\langle T_l, T_{l'}\rangle$ *is 3, and* $\langle T_l, T_{l'}\rangle$ *is orthogonal to* $-\mathrm{i}|\psi\rangle$ *and to* $A_j|\psi\rangle, B_j|\psi\rangle, C_j|\psi\rangle$ *for all* $j \notin \{l, l'\}$.

Proof. We only need to prove that $B_l|\psi\rangle = -B_{l'}|\psi\rangle$. The rest of the statement follows from lemma 2.4. Equation (1) and the hypothesis $A_l|\psi\rangle = A_{l'}|\psi\rangle$ imply that $(-1)^{i_l} c_I = (-1)^{i_{l'}} c_I$, so if $c_I \neq 0$ then $i_l = i_{l'} \bmod 2$. It follows that if $c_{I_l} \neq 0$ then $i_l = i_{l'} + 1 \bmod 2$. Equation (3) and the hypothesis $C_l|\psi\rangle = C_{l'}|\psi\rangle$ imply that $c_{I_l} = c_{I_{l'}}$ for all I. Hence, for all I we have

$$\langle I|B_l|\psi\rangle = (-1)^{i_l} c_{I_l} = (-1)^{i_{l'}+1} c_{I_{l'}} = -\langle I|B_{l'}|\psi\rangle.$$

This establishes the claim. □

The strengthened lemma 3.3 yields the following strengthened version of proposition 2.5. We omit the proof because it requires only a minor change to apply 3.3 in the proof of proposition 2.5.

Proposition 3.4. *Strengthened version of proposition 2.5. Let* $|\psi\rangle$ *be a state vector and let* M *be its associated matrix. Suppose that* $\dim\langle T_l, T_{l'}\rangle \leqslant 4$ *for some* $1 \leqslant l < l' \leqslant n$. *Then* $|\psi\rangle$ *is local unitary equivalent to a state vector* $|\psi'\rangle$ *such that* $A_l|\psi'\rangle = A_{l'}|\psi'\rangle$,

$B_l|\psi'\rangle = -B_{l'}|\psi'\rangle$, $C_l|\psi'\rangle = C_{l'}|\psi'\rangle$, the dimension of $\langle T_l, T_{l'}\rangle$ is 3, and $\langle T_l, T_{l'}\rangle$ is orthogonal to $-\mathrm{i}|\psi\rangle$ and to $A_j|\psi\rangle$, $B_j|\psi\rangle$, $C_j|\psi\rangle$ for all $j \notin \{l, l'\}$.

Next, we state and prove a stronger version of proposition 2.6.

Proposition 3.5. *Minimum rank for submatrices of M. Let $\mathcal{S} = T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_q} \cup \{-\mathrm{i}|\psi\rangle\}$ be a union of q triples together with the rightmost column $-\mathrm{i}|\psi\rangle$ of M. Then*

$$\dim\langle\mathcal{S}\rangle \geqslant \begin{cases} \dfrac{3q}{2} + 1 & q \text{ even} \\ \dfrac{3q+1}{2} + 1 & q \text{ odd.} \end{cases}$$

Proof. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be a union of some number p of triples, maximal with respect to the property that $\langle\mathcal{S}_0\rangle$ contains a subspace W for which

(i) $\dim W \geqslant \begin{cases} \dfrac{3p}{2} & p \text{ even} \\ \dfrac{3p+1}{2} & p \text{ odd} \end{cases}$, and

(ii) $W \perp \langle \mathcal{C}\backslash\mathcal{S}_0\rangle$.

We separate the argument into cases. We show that in every case, either proposition 3.5 holds or we can derive a contradiction by constructing a superset $\mathcal{S}_1$ such that $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}$, and $\mathcal{S}_1$ is the union of some number $p' > p$ triples and contains a subspace W' satisfying properties (i) and (ii) with p' in place of p. The construction of $\mathcal{S}_1$ violates the maximality of $\mathcal{S}_0$ and therefore rules out the case in question.

Case 1: Suppose that $p = q$. Then proposition 3.5 holds.

Case 2: Suppose that $p < q$ and that the triples $T_{j_1}, T_{j_2}, \ldots, T_{j_{q-p}}$ in $\mathcal{S}\backslash\mathcal{S}_0$ have the maximum possible span, that is,

$$\dim\left\langle T_{j_1}, T_{j_2}, \ldots, T_{j_{q-p}}\right\rangle = 3(q-p).$$

Properties (i) and (ii) imply that

$$\begin{aligned} \dim\langle\mathcal{S}\rangle &\geqslant \dim W + \dim\langle\mathcal{S}\backslash\mathcal{S}_0\rangle \\ &\geqslant \frac{3p}{2} + 3(q-p) \\ &= \frac{6q-3p}{2} \\ &\geqslant \frac{6q-(3q-3)}{2} \qquad (\text{since } p \leqslant q-1) \\ &= \frac{3q+3}{2} \\ &= \frac{3q+1}{2} + 1 \end{aligned}$$

and so proposition 3.5 holds. Note that if $p = q - 1$, the hypothesis of full span is met by proposition 2.2. Therefore, in the remaining cases we need only to consider $p \leqslant q - 2$.

Case 3: Suppose $p \leqslant q-2$ and that there is a pair of triples $T_l, T_{l'}$ in $\mathcal{S}\backslash\mathcal{S}_0$ with $\dim\langle T_l, T_{l'}\rangle \leqslant 4$. Let $\mathcal{S}_1 = \mathcal{S}_0 \cup T_l \cup T_{l'}$, let $p' = p + 2$, and let $W' = W \oplus \langle T_l \cup T_{l'}\rangle$, where '$\oplus$' denotes the orthogonal direct sum. That the sum is orthogonal is guaranteed by property (ii) for W. Proposition 2.5 implies that property (ii) also holds for the pair $(\mathcal{S}_1, W')$ and that

$\dim W' \geqslant \dim W + 3$ (in fact, we have equality by proposition 3.4). It follows that if p is even, so is p' and we have

$$\dim W' \geqslant \frac{3p}{2} + 3 = \frac{3p+6}{2} = \frac{3(p+2)}{2} = \frac{3p'}{2}$$

and similarly if p and p' are odd we have

$$\dim W' \geqslant \frac{3p+1}{2} + 3 = \frac{3p'+1}{2}$$

so $(\mathcal{S}_1, W')$ satisfies property (i). Thus $\mathcal{S}_1$ violates the maximality of $\mathcal{S}_0$, so we conclude that the hypothesis of case 3 is impossible.

Case 4: Suppose $p \leqslant q - 2$ and that there is a pair of triples $T_l, T_{l'}$ in $\mathcal{S}\backslash\mathcal{S}_0$ such that $\dim\langle T_l, T_{l'}\rangle = 5$. Applying proposition 3.2, we have four independent vectors

$$|\zeta_l\rangle, \quad |\eta_l\rangle \in \langle T_l\rangle, \qquad |\zeta_{l'}\rangle, \quad |\eta_{l'}\rangle \in \langle T_{l'}\rangle$$

orthogonal to all column vectors of M in columns outside of triples $T_l, T_{l'}$, so once again $\mathcal{S}_1 = \mathcal{S}_0 \cup T_l \cup T_{l'}$ with the subspace

$$W' = W \oplus \langle|\zeta_l\rangle, |\eta_l\rangle, |\zeta_{l'}\rangle, |\eta_{l'}\rangle\rangle$$

violates the maximality of $\mathcal{S}_0$. We conclude that the hypothesis of case 4 is impossible.

Case 5: The only remaining possibility is that $p \leqslant q - 3$. Let $\mathcal{T} = \left\{T_{j_1}, T_{j_2}, \ldots, T_{j_m}\right\}$ be a set of triples in $\mathcal{S}\backslash\mathcal{S}_0$ with $m \geqslant 3$ minimal with respect to the property

$$\dim\left\langle T_{j_1}, T_{j_2}, \ldots, T_{j_m}\right\rangle < 3m.$$

Applying proposition 2.3, we have two vectors

$$|\zeta_k\rangle, \quad |\eta_k\rangle \in \langle T_k\rangle$$

for each of the $m' \geqslant 2$ elements $k \in K$. Let

$$\mathcal{S}_1 = \mathcal{S}_0 \cup \left(\bigcup_{k\in K} T_k\right),$$

let $p' = p + m'$, and let

$$W' = W \oplus \langle\{|\zeta_k\rangle, |\eta_k\rangle\}_{k\in K}\rangle.$$

Note that property (ii) holds for $(\mathcal{S}_1, W')$. If $m' < m$, then the $2m'$ vectors in $\{|\zeta_k\rangle, |\eta_k\rangle\}_{k\in K}$ are independent by the minimality of $\mathcal{T}$, so we have

$$\dim W' \geqslant \dim W + 2m' \geqslant \frac{3p}{2} + 2m' = \frac{3p'+m'}{2} \geqslant \frac{3p'+1}{2}$$

so property (i) holds for $(\mathcal{S}_1, W')$, but this contradicts the maximality of $\mathcal{S}_0$. Finally, if $m' = m$, then $m \geqslant 4$ (since m' is even) and at least $2(m-1)$ of the vectors in $\{|\zeta_k\rangle, |\eta_k\rangle\}_{k\in K}$ must be independent, again by the minimality of $\mathcal{T}$. If p is even, then $p' = p + m$ is also even and we have

$$\dim W' \geqslant \dim W + 2(m-1) \geqslant \frac{3p}{2} + 2(m-1) = \frac{3p'+m-4}{2} \geqslant \frac{3p'}{2}.$$

If p is odd, then $p' = p + m$ is odd and we have

$$\dim W' \geqslant \dim W + 2(m-1) \geqslant \frac{3p+1}{2} + 2(m-1) = \frac{3p'+m-3}{2} \geqslant \frac{3p'+1}{2}.$$

Thus $\mathcal{S}_1$ with the subspace W' violates the maximality of $\mathcal{S}_0$. We conclude that the hypothesis of case 5 is impossible.

Having exhausted all possible cases, this completes the proof of proposition 3.5. □

Next we state and prove a general statement about additivity of ranks for bipartite states.

Proposition 3.6. *Let $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ be a state vector for a bipartite state, where $|\psi_j\rangle$ is an n_j-qubit state vector for $j = 1, 2$. Let M be the matrix associated with $|\psi\rangle$, and let M_j denote the associated matrix for $|\psi_j\rangle$ for $j = 1, 2$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the following submatrices of M:*

$$\mathcal{S}_1 = T_1 \cup T_2 \cup \cdots \cup T_{n_1} \cup \{-\mathrm{i}|\psi\rangle\}$$
$$\mathcal{S}_2 = T_{n_1+1} \cup T_{n_2+2} \cup \cdots \cup T_{n_1+n_2} \cup \{-\mathrm{i}|\psi\rangle\}$$

and let $\mathcal{B}' = T'_{j_1} \cup T'_{j_2} \cup \ldots \cup T'_{j_m}$ be a union of triples in M_1 with corresponding union $\mathcal{B} = T_{j_1} \cup T_{j_2} \cup \ldots \cup T_{j_m}$ in M. We have

(i) $\mathrm{rank}_{\mathbb{R}} M - 1 = (\mathrm{rank}_{\mathbb{R}} M_1 - 1) + (\mathrm{rank}_{\mathbb{R}} M_2 - 1)$,
(ii) $\mathrm{rank}_{\mathbb{R}} M_j = \dim\langle \mathcal{S}_j \rangle$ *for* $j = 1, 2$, *and*
(iii) $\dim\langle \mathcal{B}' \rangle = \dim\langle \mathcal{B} \rangle$.

Proof. Let x denote the state represented by $|\psi\rangle$ and let G denote the local unitary group. Let x_j denote the state represented by $|\psi_j\rangle$, and let G_j denote the local unitary group for $j = 1, 2$, so we have $G = G_1 \times G_2$.

It is easy to see that the G-orbit of x is diffeomorphic to the product of the G_j-orbits of the x_j, so dimensions add.

$$\dim Gx = \dim G_1 x_1 + \dim G_2 x_2$$

Applying proposition 2.1, it follows immediately that (i) holds.

For (ii), observe that the columns of $\mathcal{S}_j$ are simply the columns of M_j tensored with $|\psi_2\rangle$. The same reasoning applied to the subset $\mathcal{B} \subseteq \mathcal{S}_1$ establishes (iii). □

We end this section with statements about factoring singlets and unentangled qubits.

Proposition 3.7. *There are two triples $T_l, T_{l'}$ with $\dim\langle T_l, T_{l'}\rangle = 3$ if and only if the state factors as a product of a singlet in qubits l, l' times a state in the remaining qubits.*

Proof. Without loss of generality, let us renumber the qubits so that $l = 1, l' = 2$.

First we prove the 'if' part of the statement. Let $|\psi\rangle = |s\rangle \otimes |\phi\rangle$, where $|s\rangle$ is a singlet in qubits 1 and 2, and $|\phi\rangle$ is an $(n-2)$-qubit state. Then $|\psi\rangle$ is local unitary equivalent to $|\psi'\rangle = |s'\rangle \otimes |\phi\rangle$, where $|s'\rangle = |00\rangle + |11\rangle$. A simple calculation shows that the two triples in the matrix for a 2-qubit singlet state vector $|s'\rangle$ together span three dimensions. Therefore, by proposition 3.6 (iii), the dimension of the span of triples 1 and 2 in the matrix for $|\psi'\rangle$ is also 3. Since the rank of unions of triples is local unitary invariant by proposition 3.1, we conclude that the dimension of the span of triples 1 and 2 in the matrix for $|\psi\rangle$ is also 3.

Next we prove 'only if'. Let $|\psi\rangle$ be a state vector for which $\dim\langle T_1, T_2\rangle = 3$. By proposition 3.4, $|\psi\rangle$ is local unitary equivalent to the state vector $|\psi'\rangle$ for which $A_1|\psi'\rangle = A_2|\psi'\rangle$ and $C_1|\psi'\rangle = C_2|\psi'\rangle$. Equation (1) and the hypothesis $A_1|\psi\rangle = A_2|\psi\rangle$ imply that $(-1)^{i_1} c_I = (-1)^{i_2} c_I$, so if $c_I \neq 0$ then $i_1 = i_2 \bmod 2$, so every I for which $c_I \neq 0$ has either both zeros or ones in the first two indices. Equation (3) and the hypothesis $C_1|\psi\rangle = C_2|\psi\rangle$ imply that $c_{I_1} = c_{I_2}$ for all I. Apply this to I for which $i_1 \neq i_2$ and we get

$$c_{(00 i_3 i_4 \ldots i_n)} = c_{(11 i_3 i_4 \ldots i_n)}$$

for all $(i_3 i_4 \ldots i_n)$. It follows that $|\psi'\rangle$ factors as a product

$$|\psi'\rangle = (|00\rangle + |11\rangle) \otimes |\phi\rangle,$$

where $|\phi\rangle$ is an $(n-2)$-qubit state. Therefore, $|\psi\rangle$ is a product of a singlet state in the first two qubits times a state in the remaining qubits. □

Lemma 3.8. *If $A_j|\psi\rangle = \mathrm{i}|\psi\rangle$ then the jth qubit is unentangled.*

Proof. Renumber the qubits, if necessary, so that $j = 1$. By (1), the hypothesis $A_1|\psi\rangle = \mathrm{i}|\psi\rangle$ implies that if $c_I \neq 0$ then $i_1 = 0$. Therefore, $|\psi\rangle$ factors as a product

$$|\psi\rangle = |0\rangle \otimes |\phi\rangle,$$

where $|\phi\rangle$ is an $(n-1)$-qubit state vector. □

Proposition 3.9. *If the dimension of the span of a triple together with the rightmost column $-\mathrm{i}|\psi\rangle$ is 3, then the state factors as an unentangled qubit times a state in the remaining qubits.*

Proof. Let T_j be a triple such that $\dim\langle T_j \cup \{-\mathrm{i}|\psi\rangle\}\rangle = 3$. Since $\mathrm{i}|\psi\rangle$ lies in the span of $\langle T_j\rangle$, we may write

$$\mathrm{i}|\psi\rangle = \alpha A_j|\psi\rangle + \beta B_j|\psi\rangle + \gamma C_j|\psi\rangle$$

for some real α, β, γ. Choose $R \in SO(su(2))$ such that

$$R(A) = \alpha A + \beta B + \gamma C.$$

Since the adjoint representation $Ad: SU(2) \to SO(su(2))$ is surjective, we can choose $U_j \in SU(2)$ such that $\mathrm{Ad}\big(U_j^\dagger\big) = R$, that is, $U_j^\dagger X U_j = R(X)$ for all $X \in su(2)$. For $1 \leqslant k \leqslant n$, $k \neq j$, set U_k equal to the identity. Finally, let $U \in G = \mathrm{SU}(2)^n$ be $U = \prod_{i=1}^n U_i$. We have

$$U^\dagger A_j U|\psi\rangle = \mathrm{i}|\psi\rangle.$$

Applying U to both sides, we obtain

$$A_j U|\psi\rangle = \mathrm{i}U|\psi\rangle.$$

Applying lemma 3.8 to the matrix M' for the state $U|\psi\rangle$ shows that the jth qubit is unentangled for the state $U|\psi\rangle$. Since unentanglement of a particular qubit is local unitary invariant, the proposition is established. □

Proposition 3.10. *Rank of unentangled triples: If a state has k unentangled qubits, the rank of the union of the triples corresponding to those qubits together with the rightmost column $-\mathrm{i}|\psi\rangle$ of M is $2k+1$.*

Proof. Let $|\psi\rangle$ be a state vector for a state with k unentangled qubits, and let us renumber the qubits, if necessary, so that the unentangled qubits are numbered 1 through k. The state vector $|\psi\rangle$ is local unitary equivalent to a state vector

$$|\psi'\rangle = |00\cdots 0\rangle \otimes |\phi\rangle,$$

where $|00\cdots 0\rangle$ is the product of k unentangled qubits and $|\phi\rangle$ is an $(n-k)$-qubit state. Let M' be the matrix associated with $|\psi'\rangle$, M_1 the matrix for $|00\cdots 0\rangle$, and M'' the matrix for the single qubit state $|0\rangle$. Apply proposition 3.6 (i) to $|00\cdots 0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle$ to get $\mathrm{rank}_{\mathbb{R}} M_1 = 2k+1$ (using the fact that the single qubit state $|0\rangle$ has $\mathrm{rank}_{\mathbb{R}} M'' = 3$). Then apply proposition 3.6 (ii) to the set $\mathcal{S}_1$ which is the union of the first k triples of the matrix

M' together with the column $-\mathrm{i}|\psi'\rangle$ to get $\dim\langle\mathcal{S}_1\rangle = \mathrm{rank}_{\mathbb{R}} M_1 = 2k+1$. Finally, apply proposition 3.1 to conclude that the desired statement holds for $|\psi\rangle$. □

4. Minimum dimension orbit classification

Now we prove that any state with a minimum orbit dimension is a product of singlets for n even, times an unentangled qubit for n odd.

Main Lemma 4.1. *For $n \geqslant 2$, if M has minimum rank, then there is some pair of triples whose span is three-dimensional.*

Proof. Suppose not. A consequence of proposition 3.4 is that every pair of triples spans either three, five or six dimensions. We can rule out the possibility that some pair of triples spans five dimensions, as follows. If there is a pair $T_l, T_{l'}$ of triples which spans five dimensions, then by proposition 3.2, the pair contributes four independent column vectors which are orthogonal to every column vector in the set $\mathcal{S} = \mathcal{C}\backslash(T_l \cup T_{l'})$. Applying proposition 3.5 to $\mathcal{S}$ with $q = n-2$, we have

$$\mathrm{rank}\, M \geqslant 4 + \begin{cases} \dfrac{3q}{2} + 1 & n \text{ even} \\ \dfrac{3q+1}{2} + 1 & n \text{ odd} \end{cases}$$
$$= \begin{cases} \dfrac{3n+2}{2} + 1 & n \text{ even} \\ \dfrac{3n+3}{2} + 1 & n \text{ odd} \end{cases}$$

which is greater than minimum, and therefore impossible.

Thus we need to consider only the case where every pair of triples spans six dimensions. Let $\mathcal{T} = \{T_{j_1}, T_{j_2}, \ldots, T_{j_m}\}$ be a set of m triples minimal with respect to the property

$$\dim\langle T_{j_1}, T_{j_2}, \ldots, T_{j_m}\rangle < 3m$$

('minimal' means that $\mathcal{T}$ contains no proper subset of triples which satisfy the given property; thus, any subset of $m' < m$ triples of $\mathcal{T}$ has 'full' span of $3m'$ dimensions). We know such a $\mathcal{T}$ exists since the rank of M is minimum. Apply proposition 2.3 to get a subset $K \subseteq \{j_1, j_2, \ldots, j_m\}$ with some even positive number m' of elements and vectors $|\zeta_k\rangle, |\eta_k\rangle$ in T_k for $k \in K$. Let $\mathcal{T}' = \bigcup_{k\in K} T_k$ and let $\mathcal{S} = \mathcal{C}\backslash\mathcal{T}'$ be the union of the $q = n - m'$ triples not in $\mathcal{T}'$ together with the rightmost column $-\mathrm{i}|\psi\rangle$ of M.

If $m' < m$, the minimality of $\mathcal{T}$ guarantees that the $2m'$ vectors $\{|\zeta_k\rangle, |\eta_k\rangle\}_{k\in K}$ are independent, so the rank of M is at least (apply proposition 3.5 to $\mathcal{S}$ with $q = n - m'$)

$$\mathrm{rank}\, M \geqslant 2m' + \begin{cases} \dfrac{3q}{2} + 1 & n \text{ even} \\ \dfrac{3q+1}{2} + 1 & n \text{ odd} \end{cases}$$
$$= \begin{cases} \dfrac{3n+m'}{2} + 1 & n \text{ even} \\ \dfrac{3n+1+m'}{2} + 1 & n \text{ odd} \end{cases}$$

which is greater than minimum, so this case cannot occur.

If $m' = m$, the minimality of $\mathcal{T}$ guarantees that at least $2(m'-1)$ of the vectors in $\{|\zeta_k\rangle, |\eta_k\rangle\}_{k\in K}$ are independent, so the rank of M is at least

$$\operatorname{rank} M \geqslant 2(m'-1) + \begin{cases} \dfrac{3q}{2} + 1 & n \text{ even} \\ \dfrac{3q+1}{2} + 1 & n \text{ odd} \end{cases}$$
$$= \begin{cases} \dfrac{3n + m' - 4}{2} + 1 & n \text{ even} \\ \dfrac{3n + 1 + m' - 4}{2} + 1 & n \text{ odd.} \end{cases}$$

If $m' > 4$, this is greater than minimum, so we may assume that $m' = m \leqslant 4$.

We rule out the possibility $m = 2$, since every pair of triples spans six dimensions, so the only remaining case is $m' = m = 4$. We may further assume that *every* minimal set of triples that has less than full span consists of $m = 4$ triples, and that applying 2.3 to such a set yields $m' = 4$. Thus the columns of M decompose into a disjoint union

$$\mathcal{C} = \mathcal{S}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_t,$$

where each $\mathcal{T}_i$ is a union of four triples for which applying proposition 2.3 yields six independent vectors orthogonal to $\langle \mathcal{C}\backslash\mathcal{T}_i\rangle$, and $\mathcal{S}_0$ is a union of $q = n - 4t$ triples together with the rightmost column $-\mathrm{i}|\psi\rangle$ of M such that the span of the union of the triples in $\mathcal{S}_0$ is $3q$-dimensional.

We consider the cases $q \geqslant 2$, $q = 0$, and finally $q = 1$.

If $q \geqslant 2$, then we have

$$\operatorname{rank} M \geqslant 6t + 3q = 6\left(\frac{n-q}{4}\right) + 3q = \frac{3n+3q}{2}$$

which is greater than minimum, so this case cannot occur.

If $q = 0$ or $q = 1$, let $\mathcal{S}_1 = \mathcal{S}_0 \cup \mathcal{T}_1$ and consider the disjoint union

$$\mathcal{C} = \mathcal{S}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_t$$

of the set $\mathcal{S}_1$ with $t' = t - 1$ unions of four triples $\{\mathcal{T}_i\}_{i=2}^{t}$.

If $q = 0$, $\mathcal{S}_1$ has a span of at least nine dimensions, since any three of the triples in $\mathcal{T}_1$ have full span. We have

$$\operatorname{rank} M \geqslant 9 + 6t' = 9 + 6\left(\frac{n-4}{4}\right) = \frac{3n+6}{2}$$

which is greater than minimum, so this case cannot occur.

If $q = 1$, then $\mathcal{S}_1$ is the union of five triples together with the rightmost column $-\mathrm{i}|\psi\rangle$. If any subset of four of those five triples has full span, then we have

$$\operatorname{rank} M \geqslant 12 + 6t' = 12 + 6\left(\frac{n-5}{4}\right) = \frac{3n+9}{2}$$

which is greater than minimum, so it must be the case that *all* subsets of four triples have less than full span, so any subset of four triples contributes six independent vectors orthogonal to the remaining triple and the rightmost column $-\mathrm{i}|\psi\rangle$ of M. If any one of the qubits corresponding to one of the five triples is *not* unentangled, then by proposition 3.9 we have

$$\operatorname{rank} M \geqslant 6 + 4 + 6t' = 10 + 6\left(\frac{n-5}{4}\right) = \frac{3n+5}{2}$$

which is greater than minimum, so it must be the case that *all* five qubits are unentangled. But then by proposition 3.10, we have

$$\text{rank}\, M \geqslant 11 + 6t' = 11 + 6\left(\frac{n-5}{4}\right) = \frac{3n+7}{2}$$

which is greater than minimum.

Since all possible cases lead to contradictions, we conclude that some pair of triples must have a three-dimensional span. □

Corollary 4.2. *Any state which has minimum orbit dimension is a product of singlets when n is even, together with an unentangled qubit when n is odd.*

Proof. Let $|\psi\rangle$ be a state vector for a state with minimum orbit dimension, with associated matrix M. Apply lemma 4.1 to M to get a pair of triples whose span is three-dimensional. By proposition 3.7, $|\psi\rangle$ factors as a product of a singlet in those two qubits times an $(n-2)$-qubit state, say, with state vector $|\psi_1\rangle$, in the remaining qubits. Let M_1 be the matrix associated with $|\psi_1\rangle$. By proposition 3.6 (i), M_1 also has minimum rank. Repeating this reasoning yields a sequence $|\psi\rangle, |\psi_1\rangle, |\psi_2\rangle, \ldots$ which eventually exhausts all the qubits of $|\psi\rangle$ unless n is odd, in which case a single unentangled qubit remains. □

Next, we classify states with minimum orbit dimension up to local unitary equivalence.

Proposition 4.3. *Separation of singlet products. Products of singlets are local unitary equivalent if and only if the choices of entangled pairs are the same in each product.*

Proof. If $|\psi\rangle$ is a product of singlets for which some pair of qubits, say qubits 1 and 2, forms a singlet, then $|\psi\rangle$ is of the form $|\alpha\rangle \otimes |\beta\rangle$, where $|\alpha\rangle$ is a singlet in qubits 1 and 2. Clearly, any state local unitary equivalent to $|\psi\rangle$ is also of this form. □

The following classification theorem summarizes the results of this section.

Theorem 4.4. *Classification of states with minimum orbit dimension. An n-qubit pure state has minimum orbit dimension* $3n/2$ *(n is even) or* $(3n+1)/2$ *(n is odd) if and only if it is a product of singlets, together with an unentangled qubit for n odd. Furthermore, two of these states which do not have the same choices for pairs of entangled qubits are not local unitary equivalent.*

5. Conclusion

The classification of types of quantum entanglement is a difficult but central problem in the field of quantum information. Entanglement types partially distinguish themselves by their local unitary orbit and their orbit dimension. As an integer that can be readily calculated for a given quantum state, orbit dimension is a convenient LU invariant. It provides a useful 'first stratification' of the quantum state space, which suggests a two-step program for entanglement classification. The first step is to understand the possible orbit dimensions for a composite quantum system, and the second step is to understand the types of entanglement that occur in each orbit dimension. For pure states of n-qubits, the first step was achieved in [8]. The present work completes the second step for the orbits of the minimum dimension. In particular, states with minimum orbit dimension are precisely products of pairs of qubits, each pair in a singlet state (or LU equivalent). While there is much work left to be done in the second step of the classification program, it is worth remarking that the present results, dealing with an

Table 1. Corresponding equation and statement numbers in [8].

Number in this paper	Number in [8]
(1)	(10)
(2)	(11)
(3)	(12)
(4)	(13)
(5)	(1)
2.1	3.3
(6)	(14)
2.2	5.1
2.3	6.1
2.4	5.3
2.5	6.2
2.6	7.2
2.7	7.1

arbitrary number of qubits, give some hope that a meaningful classification of entanglement for n qubits is possible.

Reduced orbit dimension states appear to be the most interesting states. Carteret and Sudbery [7] pointed out that reduced orbit dimension states (which they call 'exceptional states') must have extreme values of the local unitary invariants that are used in the construction of all measures of entanglement. They concluded that reduced orbit dimension states should be expected to be particularly interesting and important. Subsequent studies have confirmed this expectation. All of the 'famous' states that theorists use for examples and that experimentalists try to exhibit in the laboratory have reduced orbit dimension (examples include the EPR singlet state, the GHZ state, unentangled states, the W state, and n-cat states). Both the most entangled states and the least entangled states have reduced orbit dimension.

Orbit dimension provides a useful first step in entanglement classification, but the numerical value of orbit dimension, beyond providing a sense of how rare an entanglement type is, does not carry a simple physical meaning. For example, in the case of pure three-qubit states [7, 9], the minimum orbit dimension is 5 and the maximum orbit dimension is 9. States with orbit dimension 5 consist only of products of a singlet pair and a qubit, orbit dimension 6 contains the unentangled states, orbit dimension 7 contains GHZ states as well as products of generic two-qubit states with an unentangled qubit, orbit dimension 8 contains the W state, and orbit dimension 9 contains all generic states.

Minimum orbit dimension states have the most symmetry with respect to local transformations in that they remain invariant under a larger class of transformations than any other states. They are maximal symmetry generalizations of the spin singlet state, which is invariant (as a quantum state, not an entanglement type) to any rotation applied identically to both spins. The present result, that the n-qubit maximal symmetry generalizations of the two-qubit singlet state are themselves products of singlets, shows a special role for the two-qubit singlet state in the theory of n-qubit quantum entanglement.

Linden, Popescu and Wootters [10, 11] have shown that almost all pure n-qubit quantum states lack essential n-qubit quantum entanglement in the sense that they can be reconstructed from their reduced density matrices. It appears likely (and is known in the three-qubit case) that states with essential n-qubit entanglement are found among the reduced orbit dimension states. The present work shows that minimum orbit dimension states do not exhibit essential n-qubit entanglement for $n \geqslant 3$. Rather, minimum orbit dimension states maximize pairwise entanglement. We conjecture that states with minimum orbit dimension among *non-product* states have essential n-qubit entanglement in the sense of [10, 11].

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Appendix. Equation and statement numbers in [8]

Table 1 gives a list of equation and statement numbers in the present paper with their matching numbers in [8].

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Gudder S 2003 Quantum computation *Am. Math. Mon.* **110** 181–201
[3] Linden N and Popescu S 1998 On multi-particle entanglement *Fortschr. Phys.* **46** 567–78 (*Preprint* quant-ph/9711016)
[4] Linden N, Popescu S and Sudbery A 1999 Non-local properties of multi-particle density matrices *Phys. Rev. Lett.* **83** 243–7 (*Preprint* quant-ph/9801076)
[5] Acín A, Andrianov A, Costa L, Jané E, Latorre J I and Tarrach R 2000 Generalized Schmidt decomposition and classification of three-quantum-bit states *Phys. Rev. Lett.* **85** 1560 (*Preprint* quant-ph/0003050)
[6] Acín A, Andrianov A, Jané E and Tarrach R 2001 Three-qubit pure-state canonical forms *J. Phys. A: Math. Gen.* **34** 6725 (*Preprint* quant-ph/0009107)
[7] Carteret H A and Sudbery A 2000 Local symmetry properties of pure 3-qubit states *J. Phys. A: Math. Gen.* **33** 4981–5002 (*Preprint* quant-ph/0001091)
[8] Lyons D W and Walck S N 2005 Minimum orbit dimension for local unitary action on n-qubit pure states *J. Math. Phys.* **46** 102106 (*Preprint* quant-ph/0503052)
[9] Walck S N, Glasbrenner J K, Lochman M H and Hilbert S A 2005 Topology of the three-qubit space of entanglement types *Phys. Rev.* A **72** 052324 (*Preprint* quant-ph/0507208)
[10] Linden N, Popescu S and Wootters W K 2002 Almost every pure state of three qubits is completely determined by its two-particle reduced density matrices *Phys. Rev. Lett.* **89** 207901
[11] Linden N and Wootters W K 2002 The parts determine the whole in a generic pure quantum state *Phys. Rev. Lett.* **89** 277906